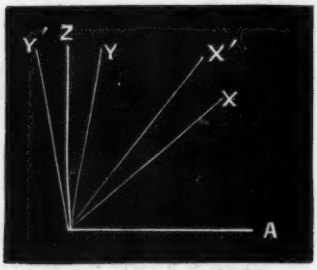
sufficed. More recently there has seemed to be a tendency to make mensuration depend upon a previous training in demonstration and thus to put it off until the student shall have made the acquaintance with the fourth, fifth, and ninth books of geometry. This is logical and might be reasonable were it not for the fact that very frequently a training in mensuration does more for one in the way of pure geometry than the training in demonstration does for mensuration. The too great insistence on logical precision has an unfortunate effect of benumbing and repelling the young mind. For this reason the very recent movement on the part of teachers from the kindergartens up to the colleges toward a separation of the constructional parts of geometry and of mensuration from the demonstrational part of the subject, allowing the latter to come later, cannot but be welcomed. From our point of view, which is that of strict logic, the possibility of constructing or measuring a figure is of secondary importance.

Another proof of the theorem that vertical angles are equal depends on the possibility of folding the angle over onto its vertical. Thus if b be folded over so as to coincide with a', a which is the continuation of a' will fall along b' which is the continuation of b, and the angles will coincide throughout. Such a method of demonstration is entirely out of the question in spherical geometry where it is necessary carefully to distinguish between congruent and symmetrical angles and triangles. The fact that an angle (a, b) is not congruent to the angle (b, a) introduces a radical difference between the presentation of spherical geometry and the usual presentations of plane geometry. Strictly speaking the distinction between (a, b) and (b, a) should be preserved in plane geometry: for it is only by going out of the plane and assuming at least some of the geometry of space that an angle can be carried into itself in such a manner as to interchange the sides. No one who has not attempted to think through the first books of plane geometry with the distinction between symmetry and congruence in mind can have any realizing appreciation of how closely the ideas are interwoven and how important the relations of symmetry really are. Even in so careful a work as Hilbert's *Grundlagen der Geometrie* the ideas are nowhere well distinguished and the statement of one of the axioms (IV_4, page 11), where the possibility of interchanging the sides of an angle is placed in a parenthesis as if it were an unimportant alternative, is a good illustration of the lack of attention which the subject receives. It is to be noted that in the usual definition of a right angle as an angle which is equal to its supplement the word "equal" undoubtedly stands for "symmetric." This will be far clearer after the definition of right angle dependent solely on congruence and capable of use on the surface of the sphere has been given.

Theorem 35. *Given an angle (a, c) there exists one and only one direction b within the angle and such that the angle (a, b) is congruent to the angle (b, c).*

This theorem shows that any angle may be bisected although no construction for the bisector is given. The proof of the theorem is not much different from that given for Theorems 3 and 4 in Lecture II. It depends on the property of continuity possessed by the directions issuing from a point. Let x be an arbi-

trary direction issuing from the vertex of the given angle. Let the angle (x, y) be taken congruent to the angle (a, x). The angle (a, y) includes the angle (a, c) or is included by it or is congruent to it so that y falls on c. Separate all the directions which issue from the vertex of the angle (a, c) and lie within the angle into two classes of which the first contains all directions such that the double angle (a, y) is included by the given angle (a, c) and the second class contains all the other directions, namely those which are such that the angle (a, y) includes the angle (a, c) or is congruent to it. It remains to show that any direction of the first class precedes any direction of the second class. Let x and x' be two directions such that the angle (a, x) includes the angle (a, x'). Let (x, y) and (x', y') be respectively congruent to (a, x) and (a, x'). Let the angle (y, z) be congruent to (a, x). Now the angle (a, z) includes the angle (a, y): for otherwise (x, y) would include (x', z) which is impossible by our hypothesis. Also (x', y') includes (x', z). Hence y' follows z and *a fortiori* follows y. Therefore if the direction x precedes the direction x', the direction y must precede y' and conversely. Applying this to the case above, it is seen that every direction in the first class must precede every direction in the second class. Hence there exists some direction, say b, such that no direction in the first class follows it and none in the second class precedes it (by the definition of continuity given in Lecture I). The direction y which corresponds to this direction b cannot either follow c or precede it. Consequently in this case c and y coincide. Hence the angles (a, b) and (b, c) are congruent and the theorem is proved.



Definition. *An angle (a, b) is said to be symmetrical to the angle (c, d) when it is congruent to the angle (d, c).*

Theorem 36. *If the angle (a, b) is symmetrical to (c, d), then conversely the angle (c, d) is symmetrical to (a, b).*

Theorem 37. *Two angles symmetrical to the same angle or to congruent angles are congruent to one another.*

These theorems follow so easily from the definition that no demonstrations will be given. It is important to notice that if two angles are symmetrical we have only to alter the order in which we name the sides of one of them and they become congruent.

Definition of right angle: *The congruent angles formed by bisecting a straight angle are called right angles.* Or, *if two symmetrical adjacent angles are such that the two corresponding sides which are not coincident form the two opposite directions of a line, the angles are said to be right angles.*

Theorem 38. *From a given point of a given line and on a given side of the line there issues one and only one direction which makes right angles with the given line.*

Theorem 39. *Any two right angles are either congruent or symmetrical.*

Theorem 40. *If two lines meet so as to form right angles on either side of either*

line, all of the four angles formed are right angles and the lines may be said to be perpendicular.

These are the familiar theorems concerning the erection of perpendiculars. The proofs will be left to the reader.

REPRESENTATION OF REAL AND IMAGINARY LOCI IN THE SAME PLANE.

By PROFESSOR G. W. GREENWOOD. McKendree College.

In elementary analytical geometry, where only the real parts of the locii are represented, it is difficult at times to give satisfactory definitions. Take for example the polar of a point with respect to a conic. If we define it as the locus of the intersection of tangents at points where the conic is cut by any secant through the point, we get when the point is without the conic a portion of the polar upon which no tangents meet; if we define it as a chord of contact, our definition appears strained when the point is within the conic.

I have been using, with some success, a method which I have not seen in any text book, and which is in brief as follows:

Consider the circle $x^2+y^2=a^2$. Taking value of x greater than a, numerically, we get $y=\pm ib$, say, where b is real. Plot the points obtained by taking the real factors in the values of y and we get the hyperbola

$$x^2-y^2=a,$$

which we may draw in red ink. This will contain all imaginary points of the circle of the form (c, ib).

Now either definition of the polar of a point on $y=0$ will apply; if the point is outside the circle, it is within the hyperbola and its polar is the locus of the intersection of real tangents to the circle or imaginary tangents (*i. e.* those to the hyperbola) at points where the locus is cut by secants through the point. When the point is within the circle, the chord of contact joins the points of tangency of the imaginary tangents.

The radical axis of two circles appears in all cases as the line through two common points, real or imaginary.

Two circles are seen to intersect in two infinite imaginary points, and two concentric circles are said to have double contact at infinity, the tangent there being the circular lines.

The limiting forms of the circle and its hyperbola illustrate graphically the dual interpretation of the equation

$$x^2+y^2=0,$$

as a point circle and a pair of lines.

If we treat the ellipse

$$x^2/a^2+y^2/b^2=1$$

in like manner we see, among other things, that the tangents from a focus are perpendicular and that their chord of contact is the corresponding directrix.

Reciprocally, the ellipse supplements the hyperbola in picturing the imaginary part.

TRIANGULAR RESIDUES.

By ORLANDO S. STETSON, Syracuse University.

The summation of arithmetical progressions which commence with unity and whose differences are 1, 2, 3, etc., leads to the theory of *polygonal numbers.*

Employing the difference 1, we obtain the *triangular numbers.* The πth one is $\frac{1}{2}\pi(\pi+1)$. If ρ be any prime number and the series of the first $\rho-2$ triangular numbers be formed, the least positive remainders (modρ) of the first $\frac{1}{2}(\rho-1)$ of these numbers will be all different. Indeed, if two of the remainders were equal for values π_1 and π_2 of π, we should have

$$(\pi_1-\pi_2)(\pi_1+\pi_2+1)\equiv 0 \pmod{\rho},$$

which is impossible. The remainder of the middle [$\frac{1}{2}(\rho-1)$th] term $\frac{1}{8}(\rho^2-1)$ occurs once only, while the remainders of the remaining terms repeat themselves in reverse order.

The $\frac{1}{2}(\rho-1)$ different remainders are called *triangular residues* and the remaining numbers of the series 1, 2, ..., $\rho-1$, are called *triangular non-residues.*

Let λ be the least positive remainder of the middle term $\frac{1}{8}(\rho^2-1)$. Then

$$8\lambda+1\equiv 0 \pmod{\rho}. \tag{1}$$

Theorem I.* *If ρ be any prime number excepting 3, the sum of the triangular residues* $\equiv\mu \pmod{\rho}$, *where* $2\mu\equiv\lambda \pmod{\rho}$.

If $f(\pi)$ denote the sum of $1+3+6+\ldots\ldots\frac{\pi(\pi+1)}{2}$, then $S_t\equiv f\left(\frac{\rho-1}{2}\right)$, where S_t denotes the sum of all the triangular residues. Now,

$$2f(\pi)=2\sum_{h=1}^{\pi}\tfrac{1}{2}h(h+1)=\sum_{h=1}^{\pi}(h^2+h).$$

Summing the series and multiplying both sides of the congruence by 3, we have

*Proved first by M. Stern, *Journal fur Mathematik*, Bd. 69.

$$\pi(\pi+1)(\pi+2)\equiv 6(t_1+t_2+\ldots+t_\pi) \pmod{\rho},$$

t_i denoting a triangular residue.

Putting $\pi=\frac{1}{2}(\rho-1)$, and multiplying both sides by 8, we have

$$3(-1)\equiv 48(t_1+t_2+\ldots+t_{\frac{1}{2}(\rho-1)}) \pmod{\rho}.$$

Hence in all cases, excepting $\rho=3$, we have

$$-1\equiv 16(t_1+t_2+\ldots+t_{\frac{1}{2}(\rho-1)}) \pmod{\rho}. \tag{2}$$

Applying (1), it follows that

$$t_1+t_2+\ldots+t_{\frac{1}{2}(\rho-1)} \equiv \mu \pmod{\rho}.$$

In view of (2), we have

Theorem II. *The sum of the triangular residues is a quadratic residue or non-residue according as ρ is of the form $4m+1$ or $4m+3$.*

Theorem III. *The sum of the triangular non-residues is always a quadratic residue.*

If S_t denotes the sum of the triangular residues and N_t the sum of the triangular non-residues, it follows that

$$S_t+N_t\equiv 1+2+\ldots+\rho-1, \quad S_t\equiv -N_t \pmod{\rho} \tag{3}$$

and the truth of the theorem follows from (2).

A *bi*-triangular non-residue being defined as the double of a triangular non-residue modulo ρ, we have

Theorem IV. *The sum of all the integers between the limits $\rho/2$ and ρ is congruent modulo ρ (a prime number) to the sum of the bi-triangular non residaes of ρ.*

We have

$$\sum_{\frac{1}{2}\rho+1}^{\rho-1} h = \frac{\rho(\rho-1)}{2} - \frac{\rho^2-1}{8},$$

and as a result of the definition of a bi-triangular non-residue and of Theorem I, and (3),

$$N_b\equiv -\lambda \pmod{\rho},$$

N_b denoting the sum of the bi-triangular non-residue of ρ. From the two results we obtain

$$\sum_{\frac{1}{2}\rho+1}^{\rho-1} h \equiv N_b \pmod{\rho}.$$

A METHOD OF TRANSVECTION IN THE ACTUAL COEFFICIENTS, AND AN APPLICATION TO EVECTANTS.

By O. E. GLENN, University of Pennsylvania.

[Concluded from the April Number.]

THE EVECTANT OPERATORS E AND $E^d I$.

Consider a form f and a mixed concomitant ψ,

$$f=a_0x_1^n+na_1x_1^{n-1}x_2+\ldots\ldots,\quad \psi=\xi^n x_1^n+n\xi^{n-1}\eta x_1^{n-1}x_2+\ldots\ldots$$

Any invariant of f will still maintain its invariant properties if for f there be substituted $f+k\psi$. Let $I(a_0, a_1, \ldots\ldots)$ be the invariant of f. Then

$$I_1(a_0+k\xi^n,\ a_1+k\xi^{n-1}\eta,\ \ldots\ldots)$$

is also an invariant function (contravariant). Expanding, we obtain

$$I_1=I+\sum_m \frac{k^m}{m!}\left(\xi^n\frac{\partial}{\partial a_0}+\xi^{n-1}\eta\frac{\partial}{\partial a_1}+\ldots\ldots\right)^m I.$$

Hence we have the invariant operator

$$E=\xi^n\frac{\partial}{\partial a_0}+\xi^{n-1}\eta\frac{\partial}{\partial a_1}+\ldots\ldots+\xi^n\frac{\partial}{\partial a_n}\equiv\sum_{s=0}^{s=n}\xi^{n-s}\eta^s\frac{\partial}{\partial a_s}.$$

It has been shown that the most general form of invariant is

$$I=\binom{n}{\lambda^{(h)}}\binom{y}{\lambda^{(h-1)}}\ldots\ldots\binom{r}{\lambda}_{kh-l'_h}I_{l'_hl''_h}I_{kh-l''_h}\ldots\ldots{}_{k_1-l'_1}I_{l'_1l'_2}I_{k_1-l''_1}a_{\lambda^{(h)}}a_{\mu^{(h-1)}+\lambda^{(h-1)}}\ldots\ldots a_{\mu+\lambda}a_{r+\nu-\lambda}.$$

Hence the general form of contravariant is

$$EI=S_\lambda\left(a_{\mu^{(h-1)}+\lambda^{(h-1)}}a_{\mu^{(h-2}+\lambda^{(h-2)}}\ldots a_{r+\nu-\lambda}\xi^{n-\lambda^{(h)}}\eta^{\lambda^{(h)}}+\ldots\ldots+a_{\lambda^{(h)}}\ldots\ldots a_{\mu+\lambda}\times\xi^{n+\lambda-r-\nu}\eta^{r+\nu-\lambda}\right).$$

Then, since $\partial/\partial x_1$ and $\partial/\partial x_2$ are contragredient to x_1, x_2, these may replace ξ and η in EI. Therefore,

$$E^d I=S_\lambda\left(a_{\mu^{(h-1)}+\lambda^{(h-1)}}\ldots\ldots a_{\mu+\lambda}a_{r+\nu-\lambda}\frac{\partial^n}{\partial x_1^{n-\lambda^{(h)}}\partial x_2^{\lambda^{(h)}}}+\ldots\ldots\right.$$

$$+a_{\lambda(h)}a_{\mu(h-1)+\lambda(h-1)}\ldots\ldots a_{\mu+\lambda}\frac{\partial^n}{\partial x_1^{n-r-\nu+\lambda}\partial x_2^{r+\nu-\lambda}})$$

is an invariant operator

Let $f=a_x{}^4=(\lambda^4)a_\lambda x_1{}^{4-\lambda}x_2{}^\lambda$.

$$\therefore H=(f,f)_2=(\lambda^2)\sum_{j=0}^{j=4}i_1I_{i_2j-i_1}I_{j-i_2}a_{\mu+\lambda}a_{2+\nu-\lambda}x_1{}^{4-j}x_2{}^j,$$

$$T=(f,(f,f)_3)_1=(\lambda^1)(\lambda^2)\sum_{j=0}^{j=6}i_1I_{i_2j-i_1}I_{j-i_2;k_1-l_1}I_{l_1,l_1}I_{k_1-l_1},$$
$$\times a_{\mu'+\lambda'}a_{\mu+\lambda}a_{2+\nu-\lambda}x_1{}^{6-j}x_2{}^j,$$

$$\mathbf{i}=(f,f)_4=(\lambda^4)a_\lambda a_{4-\lambda},$$

$$J=(f,(f,f)_2)_4=(\lambda^4)(\lambda^2)_{k_1-l_1}I_{l_1,l_1}I_{k_1-l_1}a_{\lambda'}a_{\mu+\lambda}a_{2+\nu-\lambda},$$

$$E\mathbf{i}=(\lambda^4)(a_{4-\lambda}\xi^{4-\lambda}\eta^\lambda+a_\lambda\xi^\lambda\eta^{4-\lambda})=2[(a_4\xi^4+a_0\eta^4)-4(a_3\xi^3\eta+a_1\xi\eta^3)$$
$$+3a_2\xi^2\eta^2],$$

$$EJ=S_\lambda(a_{\mu+\lambda}a_{2+\nu-\lambda}\xi^{4-\lambda'}\eta^{\lambda'}+a_{\lambda'}a_{2+\nu-\lambda}\xi^{4-\mu-\lambda}\eta^{\mu+\lambda}+a_{\lambda'}a_{\mu+\lambda}\xi^{2+\lambda-\nu}\eta^{2+\nu-\lambda})$$
$$=6[(a_2a_4-a_3^2)\xi^4+2(a_2a_3-a_1a_4)\xi^3\eta+(a_0a_4+2a_1a_3-3a_2^2)\xi^2\eta^2$$
$$+2(a_1a_2-a_0a_3)\xi\eta^3+(a_0a_2-a_1^2)\eta^4],$$

$$E^d\,\mathbf{i}=\binom{4}{\lambda}\left(a_{4-\lambda}\frac{\partial^4}{\partial x_1{}^{4-\lambda}\partial x_2{}^\lambda}+a_\lambda\frac{\partial^4}{\partial x_1{}^\lambda\partial x_2{}^{4-\lambda}}\right),$$

$$E^d\,J=S_\lambda(a_{\mu+\lambda}a_{2+\nu-\lambda}\frac{\partial^4}{\partial x_1{}^{4-\lambda'}\partial x_2{}^{\lambda'}}+a_{\lambda'}a_{2+\nu-\lambda}\frac{\partial^4}{\partial x_1{}^{4-\mu-\lambda}\partial x_2{}^{\mu+\lambda}}$$
$$+a_{\lambda'}a_{\mu+\lambda}\frac{\partial^4}{\partial x_1{}^{2+\lambda-\nu}\partial x_2{}^{2+\nu-\lambda}}).$$

Operating upon f, T, and H, we have,

$$(E^d\,\mathbf{i})f=(4-\lambda)!\lambda!\binom{4}{\lambda}\binom{4}{\lambda'}(2a_{4-\lambda}a_\lambda)=48i,$$

$$(E^d\,J)f=(4-\lambda)!\,\lambda!S_\lambda(\lambda^4)(3a_{\lambda'}a_{\mu+\lambda}a_{2+\nu-\lambda})=72J,$$

$$(E^d\,i)H=-2(\lambda^4)S_\lambda a_{\mu+\lambda}a_{2+\nu-\lambda}a_{4-j}=48J,$$

$$(E^d\ J)H=3S_\lambda\ S_{\lambda'}\ a_{\mu+\lambda}\ a_{2+\nu+\lambda}\ a_{\mu'-\lambda'}a_{2+\nu'-\lambda'}=36[(\tbinom{4}{\lambda})a_\lambda\ a_{4-\lambda}]^2=36i^2,$$

$$(E^d\ i)T=2(\tbinom{4}{\lambda})S_\lambda\ a_{\mu'+\lambda'}\ a_{\mu+\lambda}\ a_{2+\nu-\lambda}\ a_{4-\lambda}\ x_1^{2-j'+\lambda}\ x_2^{j'-\lambda}\equiv 0,$$

$$(E^d\ J)T=3S_\lambda\ S_{\lambda'}\ a_{\mu'+\lambda'}\ a_{\mu+\lambda}\ a_{2+\nu-\lambda}\ a_{\mu+\lambda}\ a_{2+\nu-\lambda}\ x_1^{2+\lambda'-j'}\ x_2^{j'-\lambda'}\equiv 0.$$

In general, we have the theorem:

If I is an invariant of degree $(h+2)$ of an n^{ic}, $f=0$, then the following relations exist:

$$(E^d\ I)f=(h+2)\ n!\ I,$$

$$[E^d\ (f,f)_n](f\ (f,f)_r)_s=k(f,\ (f,\ (f,f)_r)_s)_n \quad (n \text{ even}),$$

$$[E^d\ (f,\ (f,f)_t)_{2(n-t)}\ ](f\ (f,f)_r)_s=k[(f,f)_t,\ (f,\ (f,f)_r)_s)_n] \ (n \text{ even}).$$

We next prove the following theorem:

*The nth transvectant of a form over an evectant of an invariant of the form is a numerical multiple of the invariant.**

In proof, consider an invariant

$$I=S_\lambda\ (a_{\lambda\ (h)}\ a_{\mu\ (h-1)+\lambda^{(h-1)}}.....a_{\mu+\lambda}a_{r+\nu-\lambda}).$$

Then $EI=S_\lambda\ (a_{\mu\ (h-1)+\lambda^{(h-1)}}.....a_{\mu+\lambda}a_{r+\nu-\lambda}\ x_1^{n-\lambda^{(h)}}\ x_2^{\lambda\ (h)}+a_{\lambda\ (h)}a_{\mu\ (h-2)+\lambda^{(h-1)}}.....$

$$.....a_{r+\nu-\lambda}\ x_1^{n-\mu-\lambda^{(h-1)(h-1)}}\ x_2^{\mu+\lambda^{(h-1)(h-1)}}+.....).$$

This is the sum of $(h+2)$ covariants, each of order n and degree $(h+1)$. The effect of taking the nth transvectant of f over EI is to introduce the multiplier $(\tbinom{n}{t})$ and the factor a_t, [see (3)]. It is evident moreover that in the first covariant t may be replaced by $\lambda^{(h)}$, in the second by $(\mu^{(h-1)}+\lambda^{(h-1)})$, etc. Hence the result is a sum of $(h+2)$ invariants, each equal to I. Therefore

$$(f,\ EI)_n=(h+2)\text{I, or } (E^d\ I)f=n!(f,\ EI)_n.$$

The System for the Binary $7-ic$.

The complete system of concomitants of the binary septic has been exhibited by Von Gall in *Math. Annalen*, Bd. 31, in symbolic form. Following is the derivation of the base covariants, in actual coefficients, of degree one to eight, and the invariants of degrees four and twelve. The set contains covariants of all orders from one to eleven inclusive, and is derived as an application of the previous theory.

Degree 1.

$f=a_x{}^7=a_0x_1{}^7+7a_1x_1{}^6x_2+.....+a_7x_2{}^7$; the form itself.

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*Gordan, *Invariantentheorie.*

Degree 2.

$H_x{}^{10}$, the Hessian of f. Its seminvariant leading coefficient is $2(a_0a_2-a_1{}^2)$.

$K_x{}^6$, the fourth transvectant of f over itself. Its seminvariant leader is $2(a_0a_4-4a_1a_3+3a_2{}^2)$.

$l_x{}^2$, the sixth transvectant of f over itself. Its seminvariant leader is $2(a_0a_6-6a_1a_5+15a_2a_4-10a_2{}^2)$.

Degree 3.

$\gamma_x{}^{11}$, the Jacobian of f and K, with seminvariant leader $\frac{2}{5}a_0{}^2a_6-\frac{12}{5}a_0a_1a_5+6a_1{}^2a_4-12a_1a_2a_3+2a_0a_3{}^2+6a_2{}^3$.

$r_x{}^3$, the fifth transvectant of f and K. Seminvariant leading coefficient: $2a_0a_2a_7-7a_0a_3a_6+5a_0a_4a_5-2a_1{}^2a_7+7a_1a_2a_6+22a_1a_3a_5-25a_1a_4{}^2-27a_2{}^2a_5+45a_2a_3a_4-20a_3{}^3$.

$T_x{}^{15}$, the Jacobian of f and H. Seminvariant leader: $a_6{}^2a_3-3a_0a_1a_2+2a_1{}^3$.

Degree 4.

$\triangle_x{}^8$, the Hessian of K. Seminvariant leader: $\frac{8}{5}a_0{}^2a_4a_6-8a_0a_2a_4{}^2+8a_0a_3{}^2a_4-\frac{32}{5}a_0a_1a_3a_6+\frac{32}{5}a_1{}^2a_3a_5-32a_1a_3{}^3+\frac{24}{5}a_0a_2{}^2a_6-\frac{24}{5}a_1a_2{}^2a_5-24a_2{}^3a_4-2a_0{}^2a_5{}^2-18a_1{}^2a_4{}^2-8a_0a_2a_3a_5+56a_1a_2a_3a_4+\frac{52}{5}a_0a_1a_4a_5+16a_2{}^2a_3{}^2$.

$p_x{}^4$, the fourth transvectant of K over itself. Seminvariant leading term: $2(2a_0a_1a_4a_7-\frac{84}{15}a_0a_2a_4a_6-10a_0a_3a_4a_5+4a_0a_4{}^3-\frac{48}{15}a_1{}^2a_3a_7-\frac{12}{15}a_1a_2a_3a_6+\frac{168}{15}a_1a_3{}^2a_5+\frac{36}{15}a_1a_2{}^2a_7-\frac{36}{15}a_2{}^3a_6+12a_2{}^2a_4{}^2-\frac{6}{15}a_0{}^2a_5a_7-\frac{74}{25}a_0a_1a_5a_6+\frac{126}{15}a_0a_2a_5{}^2+6a_1{}^2a_4a_6+2a_1a_3a_4{}^2+\frac{72}{15}a_2{}^2a_3a_5-\frac{12}{15}a_0a_2a_3a_7-36a_2a_3{}^2a_4+\frac{1088}{225}a_0{}^2a_6{}^2+\frac{1088}{225}a_1{}^2a_5{}^2+\frac{72}{15}a_0a_3{}^2a_6-\frac{306}{15}a_1a_2a_4a_5+12a_3{}^4)$.

$s_x{}^{14}$, the Jacobian of H and K. Seminvariant leading coefficient: $2(a_0{}^2a_2a_5-2a_0a_1a_2a_4-a_0a_1{}^2a_5-6a_1{}^2a_2a_3-a_0{}^2a_3a_4+4a_0a_1a_3{}^2-a_0a_2{}^2a_3+3a_1a_2{}^3+3a_1{}^3a_4)$.

Degree 5.

$\eta_x{}^7$, the Jacobian of r and K, with seminvariant leader: $2a_0{}^2a_2a_5a_7-7a_0{}^2a_3a_5a_6-5a_0{}^2a_4a_5{}^2-2a_0a_1{}^2a_5a_7+7a_0a_1a_2a_5a_6+62a_0a_1a_3a_5{}^2-7a_0a_1a_4{}^2a_5-57a_0a_2{}^2a_5{}^2+57a_0a_2a_3a_4a_5-20a_0a_3{}^3a_5-4a_0a_1a_2a_4a_7-47a_0a_1a_3a_4a_6-15a_0a_1a_4{}^2a_5+6a_1{}^3a_4a_7-21a_1{}^2a_2a_4a_6-138a_1{}^2a_3a_4a_5+75a_1{}^2a_4{}^3+135a_1a_2{}^2a_4a_5-105a_1a_2a_3a_4{}^2+60a_1a_3{}^3a_4-2a_0a_2{}^2a_3a_7-14a_0a_2a_3{}^2a_6-12a_1{}^2a_2a_3a_7-78a_1a_2{}^2a_3a_6+36a_1a_2a_3{}^2a_5-54a_2{}^3a_3a_5+120a_2{}^2a_3{}^2a_4-40a_2a_3{}^4+44a_0a_2{}^2a_4a_6-20a_0a_2a_4{}^3+10a_0a_3{}^2a_4{}^2-2a_0{}^2a_3a_4a_7+12a_0{}^2a_4{}^2a_6+80a_1{}^2a_3{}^2a_6-40a_1a_3{}^3a_4+8a_0a_1a_3{}^2a_7+6a_1a_2{}^3a_7+24a_2{}^4a_6+6a_2{}^3a_3a_5-60a_2{}^3a_3a_5$.

$g_x{}^5$, second transvectant of K over r. Seminvariant leading coefficient: $(a_0a_4-4a_1a_3+3a_2{}^2)(12a_1a_3a_7-20a_1a_4a_6+8a_1a_5{}^2+18a_2a_3a_6-10a_2{}^2a_7+2a_2a_4a_5-20a_3{}^2a_5+10a_3a_4{}^2-2a_0a_4a_7+2a_0a_5a_6)-\frac{2}{3}(a_0a_5-3a_1a_4+2a_2a_3)(30a_1a_3a_6-3a_1a_2a_7-27a_1a_4a_5-12a_2{}^2a_6-3a_2a_3a_5+30a_2a_4{}^2-15a_3{}^2a_4+3a_0a_3a_7-18a_0a_4a_6+15a_0a_5{}^2)+\frac{2}{5}(a_0a_6-a_1a_5-5a_2a_4+5a_3{}^2)(2a_0a_2a_7-7a_0a_3a_6+5a_0a_4a_5-2a_1{}^2a_7+7a_1a_2a_6+22a_1a_3a_5-25a_1a_4{}^2-27a_2{}^2a_5+45a_2a_3a_4-20a_3{}^3)$.

Degree 6.

$\tau_x{}^2$, the Hessian of r, with seminvariant leader: $2[(2a_0a_2a_7-7a_0a_3a_6+5a_0a_4a_5-2a_1{}^2a_7+7a_1a_2a_6+22a_1a_3a_5-25a_1a_4{}^2-27a_2{}^2a_5+45a_2a_3a_4-20a_3{}^3)(6a_1a_3a_7-10a_1a_4a_6+4a_1a_5{}^2+9a_2a_3a_6-5a_2{}^2a_7+a_2a_4a_5-10a_3{}^2a_5+5a_3a_4{}^2-a_0a_4a_7+a_0a_5a_6)-(10a_1a_3a_6-a_1a_2a_7-9a_1a_4a_5-4a_2{}^2a_6-a_2a_3a_5+10a_2a_4{}^2-5a_3{}^2a_4+a_0a_3a_7-6a_0a_4a_6+5a_0a_5{}^2)^2]$.

$\beta_x{}^8$, the Jacobian of K and p. Seminvariant leader: $2(a_0a_4-4a_1a_3+3a_2{}^2)(\frac{8}{5}a_0a_2a_4a_7-\frac{4}{10}a_0a_3a_4a_6-2a_0a_4{}^2a_5-\frac{156}{15}a_1a_2a_3a_7+26a_1a_3{}^2a_6-\frac{52}{15}a_1a_3a_4a_5+6a_2{}^3a_7-\frac{234}{15}a_2{}^2a_3a_6+\frac{306}{15}a_2{}^2a_4a_5-\frac{96}{15}a_0a_1a_5a_7-\frac{86}{15}a_0a_2a_5a_6+6a_0a_3a_5{}^2+\frac{18}{15}a_1{}^2a_4a_7+18a_1a_4{}^3-\frac{78}{5}a_1a_2a_4a_6+\frac{54}{15}a_2a_3{}^2a_5-\frac{270}{15}a_2a_3a_4{}^2+\frac{112}{150}a_0{}^2a_6a_7-\frac{6}{15}a_0a_2a_4a_7+\frac{6}{15}a_0a_3{}^2a_7+\frac{6}{15}a_0a_1a_6{}^2-\frac{8}{15}a_1{}^2a_5a_6+\frac{252}{150}a_1{}^2a_2a_5{}^2+6a_3{}^2a_4)-2(a_0a_5-3a_1a_4+2a_2a_3)(2a_0a_1a_4a_7-\frac{84}{15}a_0a_2a_4a_6-10a_0a_3a_4a_5+4a_0a_4{}^3-\frac{16}{5}a_1{}^2a_3a_7-\frac{12}{15}a_1a_2a_3a_6+\frac{168}{15}a_1a_3{}^2a_5+\frac{36}{15}a_1a_2{}^2a_7-\frac{36}{15}a_2{}^3a_6+24a_2{}^2a_4{}^2-\frac{6}{15}a_0{}^2a_5a_7-\frac{74}{25}a_0a_1a_5a_6+\frac{156}{15}a_0a_2a_5{}^2+6a_1{}^2a_4a_6+2a_1a_3a_4{}^2+\frac{72}{25}a_0{}^2a_6{}^2+\frac{12}{25}a_1{}^2a_5{}^2+\frac{24}{5}a_0a_3{}^2a_6+\frac{102}{5}a_1a_2a_4a_5+12a_3{}^4)(10a_1a_3a_6-a_1a_2a_7-9a_1a_4a_5-4a_2{}^2a_6-a_2a_3a_5+10a_2a_4{}^2-5a_3{}^2a_4+a_0a_3a_7-6a_0a_4a_6+5a_0a_5{}^2)-(\frac{8}{5}a_0a_2a_4a_7-\frac{24}{5}a_0a_3a_4a_6-2a_0a_4{}^2a_5-\frac{52}{5}a_1a_2a_3a_7+26a_1a_3{}^2a_6-\frac{176}{5}a_1a_3a_4a_5+6a_2{}^3a_7-\frac{78}{5}a_2{}^2a_3a_6+\frac{102}{5}a_2{}^2a_4a_5-\frac{32}{25}a_0a_1a_5a_7-\frac{12}{25}a_0a_2a_5a_6+6a_0a_3a_5{}^2+\frac{18}{5}a_1{}^2a_4a_7+18a_1a_4{}^3-\frac{28}{5}a_1a_2a_4a_6+\frac{18}{5}a_2a_3{}^2a_5-18a_2a_3a_4{}^2+\frac{2}{25}a_0{}^2a_6a_7-\frac{2}{5}a_0a_2a_4a_7+\frac{2}{5}a_0a_3{}^2a_7+\frac{2}{5}a_0a_1a_6{}^2-\frac{1}{5}a_1{}^2a_5a_6+\frac{42}{25}a_1a_2a_5{}^2+6a_3{}^2a_4)(2a_0a_2a_7-7a_0a_3a_6+5a_0a_4a_5-2a_1{}^2a_7+7a_1a_2a_6+22a_1a_3a_5-25a_1a_4{}^2-27a_2{}^2a_5+45a_2a_3a_4-20a_3{}^3)$.

a_x, the third transvectant of p over r. Seminvariant leading coefficient: $p_0r_3-3p_1r_2+3p_2r_1-p_3r_0$. The quartic invariant of the $7-ic$ is the discriminant of $l_x{}^2$, viz:

$C=2(14a_0a_1a_6a_7-24a_0a_2a_6{}^2+60a_0a_3a_5a_6-40a_0a_6a_4{}^2-24a_1{}^2a_5a_7+234a_1a_2a_5a_6-360a_1a_3a_5{}^2+240a_1a_4{}^2a_5+60a_1a_2a_4a_7-360a_2{}^2a_4a_6+990a_2a_3a_4a_5-600a_2a_4{}^3-40a_1a_3{}^2a_7+240a_2a_3{}^2a_6-60a_3{}^3a_5+375a_3{}^2a_4{}^2-a_0{}^2a_7{}^2-25a_1{}^2a_6{}^2-81a_2{}^2a_5{}^2-18a_0a_2a_5a_7+10a_0a_3a_4a_7-50a_1a_3a_4a_6)$.

Index = weight = 14.

The invariant of degree 12 is the discriminant of $\tau_x{}^2$, viz:

$R=4[(2a_0a_2a_7-7a_0a_3a_6+5a_0a_4a_5-2a_1{}^2a_7+7a_1a_2a_6+22a_1a_3a_5-25a_1a_4{}^2-27a_2{}^2a_5+45a_2a_3a_4-20a_3{}^3)(6a_1a_3a_7-10a_1a_4a_6+4a_1a_5{}^2+9a_2a_3a_6-5a_2{}^2a_7+a_2a_4a_5-10a_3{}^2a_5+5a_3a_4{}^2-a_0a_4a_7+a_0a_5a_6)-(10a_1a_3a_6-1a_1a_2a_7-9a_1a_4a_5-4a_2{}^2a_6-a_2a_3a_5+10a_2a_4{}^2-5a_3{}^2a_4+a_0a_3a_7-6a_0a_4a_6+5a_0a_5{}^2)^2]\times[(10a_1a_3a_6-a_1a_2a_7-9a_1a_4a_5-4a_2{}^2a_6-a_2a_3a_5+10a_2a_4{}^2-5a_3{}^2a_4+a_0a_3a_7-6a_0a_4a_6+5a_0a_5{}^2)(7a_1a_4a_7-7a_1a_5a_6-5a_2a_3a_7-22a_2a_4a_6+27a_2a_5{}^2+25a_3{}^2a_6-45a_3a_4a_5-2a_0a_5a_7+2a_0a_6{}^2+20a_4{}^3)-(6a_1a_3a_7-10a_1a_4a_6+4a_1a_5{}^2+9a_2a_3a_6-5a_2{}^2a_7+a_2a_4a_5-10a_3{}^2a_5+5a_3a_4{}^2-a_0a_4a_7+a_0a_5a_6)^2]-[(2a_0a_2a_7-7a_0a_3a_6+5a_0a_4a_5-2a_1{}^2a_7+7a_1a_2a_6+22a_1a_3a_5-25a_1a_4{}^2-27a_2{}^2a_5+45a_2a_3a_4-20a_3{}^3)(7a_1a_4a_7-7a_1a_5a_6-5a_2a_3a_7-22a_2a_4a_6+27a_2a_5{}^2+25a_3{}^2a_6-45a_3a_4a_5-2a_0a_5a_7+2a_0a_6{}^2+20a_4{}^3)-(10a_1a_3a_6-a_1a_2a_7-9a_1a_4a_5-4a_2{}^2a_6-a_2a_3a_5+10a_2a_4{}^2-5a_3{}^2a_4+a_0a_3a_7-6a_0a_4a_6+5a_0a_5{}^2)(6a_1a_3a_7-10a_1a_4a_6+4a_1a_5{}^2+9a_2a_3a_6-5a_2{}^2a_7+a_2a_4a_5-10a_3{}^2a_5+5a_3a_4{}^2-a_0a_4a_7+a_0a_5a_6)]^2$.

Index = weight = 42.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

192. Also solved by J. Scheffer, Hagerstown, Md.

193. Proposed by SAUL EPSTEEN, Ph. D., Chicago, Ill.

Professor Goursat states (*Transactions of the American Mathematical Society*, January, 1904, p. 111) that if $a_1, a_2, \ldots, a_n; h_1, h_2, \ldots, h_n$ are two sequences, the h's being all positive, then $\sum \frac{a_i^2}{h_i} \geqq \frac{(\Sigma a_i)^2}{\Sigma h_i}$. Prove this.

Solution by F. L. GRIFFIN, S. B., and L. E. DICKSON, Ph. D., Chicago, Ill.

For $n=2$, the inequality becomes, upon multiplication by the *positive* number $h_1h_2(h_1+h_2)$ and transposition of terms, $(a_1h_2-a_2h_1)^2 \geqq 0$. These steps may be reversed, giving a proof for $n=2$. For the general case, we proceed by induction, assuming the formula true for $n=1, 2, \ldots, m$. Then

$$\sum_{i=1}^{m+1} \frac{a_i^2}{h_i} \geqq \frac{\left(\sum_{i=1}^{m} a_i\right)^2}{\sum_{i=1}^{m} h_i} + \frac{a^2{}_{m+1}}{h_{m+1}} \geqq \frac{\left(\sum_{i=1}^{m} a_i + a_{m+1}\right)}{\sum_{i=1}^{m} h_i + h_{m+1}} \text{ or } \frac{\left(\sum_{i=1}^{m+1} a_i\right)^2}{\sum_{i=1}^{m+1} h_i}.$$

Also solved by G. B. M. Zerr, Parsons, W. Va., for sequences where

$$a_i = a_{i-1} + m\,;\ h_i = h_{i-1} + 1.$$

195. Proposed by W. J. GREENSTREET, A. M., Editor of the Mathematical Gazette, Stroud, England.

Prove that when n is a positive integer,

$$\sum_{r=1}^{r=n} (-1)^r\, {}_nC_r\, 2^{n-r}\, r^2 = n^2 - 2n.$$

Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

$$(1+x)(1-x)^{-3} = 1 + 2^2x + 3^2x + 3^2x^2 + 4^2x^3 + \ldots\ldots (2x-1)^n$$
$$= 2^nx^n - c_1 2^{n-1}x^{n-1} + c_2 2^{n-2}x^{n-2} - \ldots\ldots$$

Required sum = coefficient of x^{n-1} in $(1+x)(1-x)^{-3}(2x-1)^n$,

i. e., in $(1+x)(1-x)^{-3}[x^n - c_1x^{n-1}(1-x) + \ldots\ldots]$,

i. e., in $(1+x)[x^n(1-x)^{-3} - c_1x^{n-1}(1-x)^{-2} + \ldots\ldots]$,

which is $2c_2 - c_1$ or $n^2 - 2n$.

Also solved by G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.

196. Proposed by L. E. NEWCOMB, Los Gatos, California.

Find the rth term of $\left(x-\frac{1}{x}\right)^n \equiv z^n$ in terms of z.

I. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathemrtics and Astronomy, McKendree College, Lebanon Ill.

Let $x=\cos\theta+i\sin\theta$, then $z=2i\sin\theta$, and $x^p=\cos p+i\sin p\theta$. This enables us to express any term of $(x-1/x)^n$ in terms of θ, *i. e.*, in terms of z.

II. Solution by F. D. POSEY, A. B., San Mateo, Cal.; G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and GRACE M. BAREIS, Bala, Pa.

$$x-1/x=z,\ \therefore x=\frac{z\pm\sqrt{(z^2+4)}}{2}.$$

The rth term of $(x-1/x)^n$ is $\frac{(-1)^{r-1}n(n-1)\ldots\ldots(n-r+2)}{(r-1)!}x^{n-r+1}\left(\frac{1}{x}\right)^{r-1}$

$$=\frac{(-1)^{r-1}n(n-1)\ldots\ldots(n-r+2)}{(r-1)!}x^{n-2r+2}$$

$$=\frac{(-1)^{r-1}n(n-1)\ldots\ldots(n-r+2)}{(r-1)!}\left(\frac{z\pm\sqrt{(z^2+4)}}{2}\right)^{n-2r+2}.*$$

197. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve $(18)^{4(2-x)}=(54\sqrt{2})^{3x-2}$.

Solution by W. W. LANDIS, Dickinson College, Carlisle, Pa.

Writing the equation in the form $(1).....18^{4(2-x)}=18^{\frac{1}{2}(3x-2}$ we find $x=\frac{2}{17}\frac{2}{3}$.†

Also solved by G. W. Greenwood, J. E. Sanders, A. H. Holmes, F. D. Posey, R. A. Wells, G. I. Hopkins, H. R. Higley, G. B. M. Zerr, E. L. Sherwood, Grace M. Bareis, J. Scheffer, L. E. Newcomb.

GEOMETRY.

219A. Proposed by H. F. MacNEISH, A.B., Assistant in Mathematics, University High School, Chicago, Ill.

Draw through a given point a line which shall divide a given quadrilateral into two equivalent parts: (1) when the point lies in a side of the quadrilateral; (2) when the point is without; (3) within the quadrilateral.

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*Dr. Zerr also gives the values

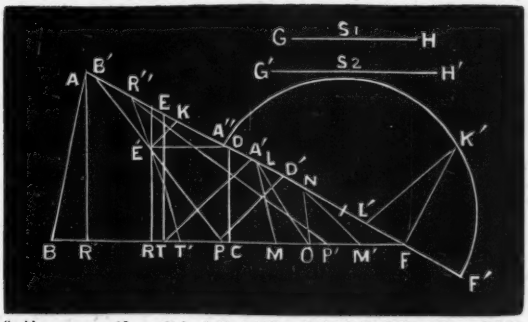
$$(-1)^{r-1}{}_nc_{r-1}(z+1/z-1/z^3+1/z^5-\ldots\ldots)^{n-2r+2},$$

$$(-1)^r{}_nc_{r-1}(1/z-1/z^3+1/z^5-\ldots\ldots)^{n-2r+2}.$$

†Owing to the periodicity of the exponential function a^y may be written $e^{y\log a+2n\pi i}$ (n=integer). It now follows from (1) that $e^{[\frac{1}{2}(22-17x)]\log 18+2n\pi i}=1$, whence $\frac{1}{2}(22-17x)\log 18+2n\pi i=0$, and therefore $x=\frac{2}{17}\frac{2}{3}+(4n\pi i/17\log 18)$. When $n=0$, $x=\frac{2}{17}\frac{2}{3}$. ED.

II. Solution by L. E. NEWCOMB, Los Gatos, California.

(1) Let $ABCD$ be the quadrilateral of area S^2, E the given point in the side AD; complete the triangle ABF and let $S_1^2=\frac{1}{2}S^2+$area DCF; $S_2^2=$area ABF $=S^2+DCF$; $GH=S_1$, $G'H'=S_2$. Draw ER and AR' perpendicular to BF, of lengths d and d', respectively. On FA lay off $FL=S_2$, $LN=S_1$, $FB'=FB$, $FA'=d'$, $FD'=d$; on FB lay off $FM=S_1$. Join LM and parallel to LM draw NO; join LO and parallel to LO draw $B'P$; join PD' and parallel to PD' draw $A'T$; join ET. The line ET divides $ABCD$ equally.



In proof, $FL:FN=FM:FO=S_2:S_1=S_1:FO$, hence $S_2.FO=S_1^2$ and therefore $S_2^2.FO=S_1^2S_2$, which may be expressed as the proportion, $S_2^2:S_1^2=S_2:FO$. Now $LF(=S_2):FO=B'F:FP$; therefore $S_2^2:S_1^2=B'F:FP$ from which follows easily, $S_2^2:\frac{1}{2}d'.B'F=S_1^2:\frac{1}{2}d'.FP$. But $B'F(=BF).\frac{1}{2}d'=S_2^2$, hence $\frac{1}{2}d'.FP=S_1^2$ and $FD'(=d):FA'(=d')=FP:FT$ and thus $\frac{1}{2}d.FT=\frac{1}{2}d'.FP$. Since $\frac{1}{2}d'.FP=S_1^2$ it follows that $\frac{1}{2}d.FT=S_1^2=\frac{1}{2}S^2+$area DCF, and consequently $\frac{1}{2}d.FT-$area $DCF=$area $ETCD=\frac{1}{2}S^2$.

(2) Let E' be the given points without the quadrilateral. For the construction of a line through E' cutting off a given area S_1^2 from the triangle ABF, see Geometry, Problem No. 218.

(3) Let E'' be the given point within the quadrilateral; draw $E''A''$ parallel to BF, EK perpendicular to AF and let $EK=a$. Extend AF, the distance $2a$, to F'. On the diameter $A''F'$ describe the semi-circle $A''K'F'$; draw FK' at right angles to AF; from the center K' with radius S_1 describe an arc cutting AF at L' and join $L'K'$. On FB lay off $FM'=a$, $FP'=S_1-L'F$; join NM', and parallel to to NM', draw $P'R''$. Through R'' and E'' draw $R''T'$ and let the altitude of the triangle $R''T'F$ to the base $R''F=b$. $R''T'$ is the required line.

In proof, $E''K(=a):b=R''A'':R''F$, hence $b.R''A''=a.R''F$; also $M'F(=a):NF(=S_1)=FP'(=S_1-L'F):R''F$, hence $a.R''F=S_1.FP'=S_1.(S_1-L'F)$. Now $L'F=\sqrt{[S_1^2-(FK')^2]}$ and $(FK')^2=2a.A''E=2a.(R''F-R''A'')$, therefore $a.R''E=S_1(S_1-\sqrt{[S_1^2-2a(R''F-R''A'')]}$, consequently $\left(\frac{a.R''F}{S_1}-S_1\right)^2=S_1^2-2a.R''F+2a.R''A''$, hence $\frac{a^2.(R''F)^2}{S_1^2}=2a.R''A''$, and $a.R''F=\frac{2R''A''.S_1^2}{R''F}$. But $a.R''F=b.R''F$, hence $b.R''A''=\frac{2R''A''.S_1^2}{R''F}$ and $b.R''F=2S_1^2$. $\therefore \frac{1}{2}b.R''F=S_1^2=\frac{1}{2}S^2+$area DCF. Consequentlp $\frac{1}{2}b.R''F-$area $DCF=\frac{1}{2}S^2=\frac{1}{2}$ of area of the quadrilateral $ABCD$.

221. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Construct a right triangle with given hypotenuse h, and having an inscribed square of side 12 with a side lying along the hypotenuse. Show further that the minimum value of h is 36, the triangle being then isosceles.

Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

Call an acute angle of the triangle a. Then $h=12\cot a+12+12\tan a=12[1+2/\sin 2a]$.

h is a minimum when $\sin 2a=1$, *i. e.*, when $a=45°$, the triangle then being isosceles, and the value of h being 36.

Also solved by S. A. Corey, A. H. Holmes, R. A. Wells, G. I. Hopkins, G. B. M. Zerr, E. L. Sherwood, J. Scheffer, L. E. Newcomb, and Grace M. Bareis.

222. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, West Va.

At the ends of a focal chord CC' of a parabola are drawn the normal chords CD, $C'D'$. Prove that DD' is parallel to CC' and equal to three times its length.

I. Solution by F. D. POSEY, A. B., San Mateo, California.

Let the equation of the parabola referred to its axis and the tangent at the vertex be $y^2=4ax$, let (h, k), (h', k') be the coördinates of the points C and C', respectively, and $y=m(x-a)$ the equation of CC'.

The tangent at C' is $k'y=2a(x+h')$ whose slope is $2a/k'$. Since CD is perpendicular to the tangent at C and since the tangents at C and C' are perpendicular to each other, CD is parallel to the tangent at C'. The equation of the normal at C has therefore the slope $2a/k'$, and since it passes through the point (h, k) its equation is $y-k=(2a/k')(x-h)$.

By solving this last equation simultaneously with $y^2=4ax$ the coördinates of the point D are found to be, after reducing,

$$\left(\frac{ah+k'^2-kk'}{a},\ 2k'-k\right)\equiv(p, q).$$

The same method gives the values of the coördinates (p', q') of D. The slope of the line through D and D' is $\frac{q'-q}{p'-p}$, which after substituting the above results reduces to $\frac{k'-k}{h'-h}=m$, since $(h, k)(h', k')$ both satisfy $y=m(x-a)$. $\therefore DD'$ is parallel to CC'.

By substituting the values obtained for p, q, p', q', in $\sqrt{[(p-p')^2+(q-q')^2]}$ the latter reduces to $3\sqrt{[(h-h')^2+(k-k')^2]}$; $\therefore DD'=3CC'$.

II. Solution by R. A. WELLS, Bellevue College, Bellevue, Neb.

Let the tangents at C, C' meet in A; denote the intersection of CD with $C'D'$ by G; draw BCE parallel to the axis of the parabola, meeting $C'D'$ in E.

The tangents CA, $C'A$ are perpendicular to each other; as also are the

chords CD, $C'D'$. $\angle CEG=\angle BCA=\angle ACC'=\angle CC'G$, hence the triangle $CC'E$ is isosceles and $C'G=GE$. Since CE is parallel to the axis of the parabola and AC is parallel to $C'D'$, E bisects $C'D'$. Hence $C'G=\frac{1}{4}C'D'$, or $C'G=\frac{1}{3}GD'$. In the same way it may be shown that $CG=\frac{1}{3}GD$. Therefore the triangles $C'CG$ and DGD' are similar. It follows easily now that CC' and $D'D$ are parallel and $CC'=\frac{1}{3}D'D$.

Also solved by G. B. M. Zerr, and J. Scheffer.

223. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Find a point C in a given line AB, so the lines joining C to the angular points of a triangle PQR coplanar with the given line may cut off on any line parallel to the given line and lying in the same plane two equal segments.

Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill., and W. W. LANDIS, Dickinson College, Carlisle, Pa.

AB cuts any one of the sides of the triangle produced, say PQ, in X. Determine the point harmonically separated from X by P and Q and call this point Y. The line joining R and Y cuts AB in the required point C. For the sheaf CX, CP, CY, CR being harmonic it is cut by any line parallel to CX in four harmonic points one of which is at infinity, therefore the remaining three points intercept equal segments.

Also solved by F. D. Posey, San Mateo, Cal.; R. A. Wells, Bellevue, Neb.; and G. B. M. Zerr, Parsons, W. Va.

CALCULUS.

176. Proposed by B. F. FINKEL, A. M., M. Sc., 204 St. Marks Square, Philadelphia, Pa.

Show by any method, Riemann's excepted, that

$$\int_0^\infty e^{-x^2}\cos\frac{b^2}{x^2}dx=\tfrac{1}{2}\sqrt{(\pi)}e^{-b\sqrt{2}}\cos b\sqrt{2}.$$

Remark by S. A. COREY, Hiteman, Iowa.

Solved by Dr. G. B. M. Zerr in his paper entitled *On the Evaluation of Certain Definite Integrals* in the MONTHLY of March, 1904, page 57, §2.

177. Proposed by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

Find the volume of the minimum cone which can be circumscribed about a hemisphere.

Solution by A. H. HOLMES, Brunswick, Maine, and S. A. COREY, Hiteman, Iowa.

Let x=altitude of cone, r=radius of sphere, R=radius of base of cone, θ=angle between x and the line joining vertex and circumference of base of cone. Then, from similar triangles, $x=r/\sin\theta$, $R=r/\cos\theta$.

Volume V of cone is, $V=\pi R^2x/3=\pi r^3/3\cos^2\theta\sin\theta$.

When V is a minimum, $dV/d\theta=0$, which condition is satisfied when $\sin\theta=\sqrt{\frac{1}{3}}$, *i. e.*, $V=\frac{1}{2}(\pi r^3\sqrt{3})$.

Also solved by R. A. Wells, J. E. Sanders, G. W. Greenwood, W. W. Landis, G. B. M. Zerr, E. L. Sherwood, J. Scheffer, and L. E. Newcomb.

MECHANICS.

167. Proposed by EDWIN S. CRAWLEY, Ph.D., Professor of Mathematics in the University of Pennsylvania.

An anchor ring or torus is produced by the revolution of a circle of radius r, the center of the revolving circle describing a circle of radius R. A quadrant of the torus is cut by two planes through the center of the ring perpendicular to each other and perpendicular to the plane of revolution. Determine the limiting value of the ratio $R:r$, so that when the quadrant thus formed is placed with one of its ends in coincidence with a horizontal plane it will rest in that position.

I. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

The center of gravity of the quadrant is at a distance d from the axis of revolution where $d=(4R^2+r^2)/\pi R\sqrt{2}$ [Routh's *Statics* (second edition), Vol. I, ex. 4, p. 280].* The limiting ratio of R to r is given by $d/\sqrt{2}=R-r$, whence $4R^2+r^2=2\pi R(R-r)$; and $R/r=2.9+$.

II. Solution by B. F. FINKEL, Fellow in Mathematics, University of Pennsylvania, Philadelphia, Pa.

The equation of the surface of the ring is $\rho=R\sin\phi\pm\sqrt{(r^2-R^2\cos^2\phi)}$, when r is the radius of a cross-section of the ring and R the radius of the path of the center of gravity of the generating circle. Then

$$\bar{x}=2.\frac{\int\sigma x\,dV}{\int dV},$$

where σ (=1, say) is the density dV and element of volume, and x the distance from the y-axis to this element of volume.

Then $dV=\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$,

$x=\rho\sin\phi\cos\theta$, and

$\int dV=2\pi R.\pi r/4=\frac{1}{4}\pi^2r^2R$. Hence

$$\bar{x}=\frac{4}{\pi^2r^2R}\int_{\cos^{-1}(r/R)}^{\frac{1}{2}\pi}\sin^2\phi\,d\phi\int_{R\sin\phi-\sqrt{(r^2-R^2\cos^2\phi)}}^{R\sin\phi+\sqrt{(r^2-R^2\cos^2\phi)}}\rho^3d\rho\int_0^{\frac{1}{2}\pi}d\theta$$

$$=\frac{8}{\pi^2r^2}\int_{\cos^{-1}(r/R)}^{\frac{1}{2}\pi}[\sin^3\phi(2R^2\sin^2\phi+r^2-R^2)\sqrt{(r^2-R^2\cos^2\phi)}]d\phi.$$

Let $\cos\phi=(r/R)\cos\psi$; then the last equation reduces to

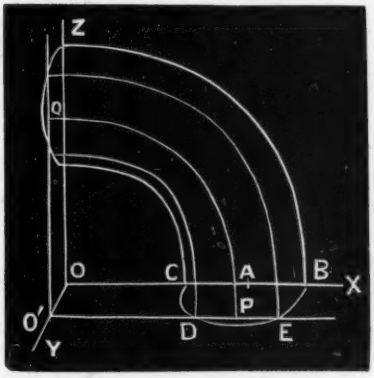
*See also Bowser's *Analytical Mechanics*, 9th ed., p. 112.

$$\bar{x}=\frac{8}{\pi^2 R}\int_0^{\frac{1}{2}\pi}[R^2+r^2-\frac{r^2}{R^2}(3R^2+r^2)\cos^2\psi+2\frac{r^2}{R^2}\cos^4\psi]\sin^2\psi d\psi=\frac{4R^2+r^2}{2\pi R}.$$

To insure stability, this must be greater than $R-r$. Hence, the limiting ratio is

$$\frac{R}{r^2}=\frac{\pi+\sqrt{(\pi^2+2\pi-4)}}{2(\pi-2)}=2.9+.$$

III. Solution by the PROPOSER.



$OA=R$, $AB=AE=r$. Equation of section $BEDC$ is

$$(x-R)^2+y^2=r^2.$$

$$\therefore O'E=R+\sqrt{(r^2-y^2)},$$
$$O'D=R-\sqrt{(r^2-y^2)}.$$

Take as element of volume the thread PQ whose length is $\frac{1}{2}\pi x$ and cross-section $dxdy$. Then, if OY be taken as axis of moments the abscissa of the center of gravity of PQ is $2x/\pi$. Hence, the moment of the attraction of gravity upon the whole solid is

$$2\int_0^r\int_{R-\sqrt{(r^2-y^2)}}^{R+\sqrt{(r^2-y^2)}}x^2dxdy=4R^2\int_0^r\sqrt{(r^2-y^2)}dy+\tfrac{4}{3}\int_0^r(r^2-y^2)^{\frac{3}{2}}dy$$

$$=4r^2R^2\int_0^{\frac{1}{2}\pi}\cos^2\theta d\theta+\tfrac{4}{3}r^4\int_0^{\frac{1}{2}\pi}\cos^4\theta d\theta=\pi r^2R^2+\tfrac{1}{4}\pi\, r^4=\frac{\pi r^2}{4}(r^2+4R^2).$$

The volume of the solid is $\frac{1}{4}\pi r^2.2\pi R=\frac{1}{2}\pi^2 r^2 R$. Hence, the arm of the center of gravity is

$$\frac{\pi r^2}{4}(r^2+4R^2)\div\tfrac{1}{2}\pi^2 r^2 R=\frac{r^2+4R^2}{2\pi R}.$$

If the solid is about to topple over the center of gravity is above C, and $\frac{r^2+4R^2}{2\pi R}=R-r$, from which

$$\frac{R}{r}=\frac{\pi\pm\sqrt{(\pi^2+2\pi-4)}}{2(\pi-2)},\text{ or discarding the negative result,}$$

$$\frac{R}{r}=\frac{\pi+\sqrt{(\pi^2+2\pi-4)}}{2(\pi-2)}=2.902,\text{ approximately.}$$

Also solved by S. A. Corey, Hiteman, Iowa.

DIOPHANTINE ANALYSIS.

120. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Find the prime numbers p for which $x^2-pxz-px-z+p^2-3=0$ has more than two sets of positive integral solutions x, z, each $<p$.

Remark by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The following sets of values satisfy the required conditions: When $x=1$, $z=p-2$; when $z=1$, $x=p-2$. When $z=x-2$: $x=\frac{1}{2}+\frac{1}{2}\sqrt{(4p+5)}$, $z=\frac{1}{2}\sqrt{(4p+5)}-\frac{3}{2}$, hence when $4p+5$ is a square there are more than two sets of values x, z, each $<p$.

$\therefore p=11$; $x=1, 4, 9$; $z=9, 2, 1$. $p=41$; $v=1, 7, 39$; $z=39, 5, 1$.
$p=19$; $x=1, 5, 17$; $z=17, 3, 1$. $p=71$; $x=1, 9, 69$; $z=69, 7, 1$.
$p=29$; $x=1, 6, 27$; $z=27, 4, 1$. $p=89$; $x=1, 10, 87$; $z=87, 8, 1$.

AVERAGE AND PROBABILITY.

150. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If the length of a circular arc be b and the radius vary uniformly, what is the average area of all the segments possible?

Remark by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Evaluating $\triangle = \underset{y \doteq 0}{\mathrm{L}} \int_y^{2\pi} \left(1 - \frac{\sin y}{y}\right)\frac{dy}{y^3} \Big/ \int_y^{2\pi} \frac{dy}{y^2}$, in my solution on p. 41, we find that,

$$\triangle = \underset{y \doteq 0}{\mathrm{L}} \frac{\pi y b^2}{2\pi - y} \int_y^{2\pi} \left(1-1+\frac{y^2}{3!}-\frac{y^4}{5!}+\frac{y^6}{7!}-\frac{y^8}{9!}+\ldots\ldots\right)\frac{dy}{y^3}$$

$$= \underset{y \doteq 0}{\mathrm{L}} \frac{\pi y b^2}{2\pi - y} \int_y^{2\pi} \left(\frac{1}{6y}-\frac{y}{5!}+\frac{y^3}{7!}-\frac{y^5}{9!}+\ldots\ldots\right)dy$$

$$= \underset{y \doteq 0}{\mathrm{L}} \frac{\pi y b^2}{2\pi - y} \left[\frac{1}{6}\log\frac{2\pi}{y} - \left(\frac{y^2}{2.5!}-\frac{y^4}{4.7!}+\frac{y^6}{6.9!}-\ldots\ldots\right)_y^{2\pi}\right]$$

$$= \underset{y \doteq 0}{\mathrm{L}} \frac{\pi y b^2}{6(2\pi - y)} \log\frac{2\pi}{y} = \underset{y \doteq 0}{\mathrm{L}} \tfrac{1}{12} b^2 y \log\frac{2\pi}{y}.$$

When $y=0$, $y\log\frac{2\pi}{y} = 0\times\infty$. But $y\log\frac{2\pi}{y} = \log\frac{2\pi}{y} \div \frac{1}{y}$; and by differentiating numerator and denominator, $\triangle = \frac{-(2\pi/y^2)/(2\pi/y)}{-(1/y^2)} = y = 0$, when $y=0$, instead of $b^2/8\pi$.

Also solved by Henry Heaton, Atlantic, Iowa, with the result $\triangle=0$.

153. Errata. For $1/6a$ read $a/6$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

200. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

No matter what value x be given, the *numerical* value of the expression $(x+2)/(2x^2+3x+6)$ can never exceed $\frac{1}{3}$.

201. Proposed by H. B. LEONARD, B. S., Graduate Student, The University of Chicago.

Solve by quadratics: $x+y+xy=75$; $x^2-y^2=315$.

GEOMETRY.

226. Proposed by W. J. GREENSTREET. A. M., Editor of The Mathematical Gazette, Stroud, England.

The triangles ABC, $A'B'C'$ are in plane perspective, and the corresponding sides BC, $B'C'$, ..., cut in P, Q, R, respectively. AA', ..., cut the line PQR in $P'Q'R'$, respectively. Show that (PP', QQ', RR') is an involution range.

227. Proposed by O. W. ANTHONY. Head of Mathematical Department, DeWitt Clinton High School, New York City.

Construct a parallelogram having given a side and the distances of its vertices from a given point.

228. Proposed by O.E. GLENN, A.M., Fellow in Mathematics, University of Pennsylvania, Philadelphia, Pa.

Given a point O without a circle S; two arbitrary lines through O cut S in the points A, A', and B, B', respectively. Prove, by pure geometry, that the four circles through OAR, OBR. $OA'R'$, $OB'R'$, respectively, intersect in points collinear with O; R and R' being points upon S arbitrarily chosen.

CALCULUS.

179. Proposed by B. F. FINKEL, A. M., M. Sc., 204 St. Marks Square, Philadelphia, Pa.

Discuss the integrals of the equation $x(1-x)w''+[1-(a+b+1)x]w'-abw=0$ in the vicinities $x=0$, and $x=1$; indicating the form for the latter vicinity when $a+b=1$. Also when $1-a-b$ is an integer l. [From Forsyth's *Linear Differential Equations*, Ex. 6, p. 103].

DIOPHANTINE ANALYSIS.

121. Proposed by L.E. DICKSON, Ph.D., Assisstant Professor in Mathematics, The University of Chicago.

Find a formula for the solutions of $x^2+y^2\equiv 1 \pmod{p}$ valid in all cases $p>2$.

AVERAGE AND PROBABILITY.

154. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Five points are taken at random in space. The chance that the fifth point may be included within the tetrahedron of which the other four points are the vertices is $\frac{1}{27}$.

GROUP THEORY.

4. Proposed by SAUL EPSTEEN, Chicago, Ill.

Find the four-parameter continuous group which leaves invariant $x_2^2 - x_1 x_3 = 0$.

NOTES.

Errata, p. 87, line 7 from bottom, middle column, for ABCEGEF, read ABECGDE.

Drs. C. M. Mason and D. R. Curtiss have been appointed to instructorships in mathematics at Yale University, the former in the Sheffield Scientific Department.

The Association of Ohio Teachers of Mathematics and Science was organized at a meeting held in Columbus April 2, 1904. The officers of the Association are: President, Charles S. Howe; Vice President, Alan Sanders; Secretary, Thos. E. McKinney.

At Stanford University Dr. H. C. Moreno has been promoted to an assistant professorship in Applied Mathematics. Professor Allardice and Mr. Manning have been granted leaves of absence for one year. Mr. Manning will spend the year studying in Europe, his work at the University being in charge of Mr. George I. Gavett.

The San Francisco Section of the American Mathematical Society met on Saturday, April 30, at Stanford University. In the morning session papers were read by Professors M. W. Haskell, H. C. Moreno, R. L. Allerdice; Dr. D. N. Lehmer, and Mr. A. W. Whitney, and in the afternoon session by Professors I. Stringham, G. A. Miller, H. F. Blichfeldt, and M. W. Haskell. The next meeting will be held on October 1, 1904.

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SPHERICAL GEOMETRY.

By EDWIN BIDWELL WILSON.

LECTURE VI. RELATIONS OF SYMMETRY.

Definition: *Two triangles ABC and $A'B'C'$ are said to be symmetrical when their corresponding sides are congruent and their corresponding angles are symmetrical.* That is, when the following congruences are satisfied

$$AB \equiv A'B', \quad BC \equiv B'C', \quad CA \equiv C'A',$$
$$\measuredangle BCA \equiv \measuredangle A'C'B', \quad \measuredangle CAB \equiv \measuredangle B'A'C', \quad \measuredangle ABC \equiv \measuredangle C'B'A',$$

Theorem 41. *Two triangles symmetrical to the same triangle or to congruent triangles are congruent to each other. If two triangles are respectively congruent and symmetrical to the same triangle or to congruent triangles they are symmetrical to each other.*

Theorem 42. *Two triangles ABC and $A'B'C'$ which have two sides of the one congruent to two sides of the other but the included angles symmetrical are symmetrical.*

Theorem 43. *Two triangles which have a side of the one congruent to the side of the other but the adjacent angles respectively symmetrical are symmetrical.*

Theorem 44. *A triangle which has two congruent sides or two symmetrical angles is symmetrical to itself.*

The first of these theorems follows directly from the congruences which by definition hold between symmetrical and between congruent triangles. To prove the second let $A^\circ B^\circ C^\circ$ be any triangle symmetrical to the given triangle ABC and let $A'B'C'$ be connected with ABC by the congruences